PAPER

Signal Reconstruction from Sampled Data Tainted by Aliasing Phenomena Using Neural Network

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Abstract **For reconstructing a signal from sampling data, the method based on the Shannon's sampling theorem is usually employed. In this method, the reconstruction error appears when the signal does not satisfy the Nyquist condition. This paper proposes a new reconstruction method by using a linear perceptron and a multilayer perceptron as the FIR filter. The perceptron which has weights obtained by learning through adapting the original signal suppresses the difference between the reconstructed signal and the original signal even when the Nyquist condition does not stand. Although the proposed method needs weight data for reconstruction, the total data size can be much smaller than with the ordinary sampling method as the most suitable reconstruction filter is exclusively adapted to the given sampling data.**

Keywords: neural network, digital signal processing, signal reconstruction, FIR filter, sampling theorem, data compression

1. Introduction

Currently, the method of reconstructing a signal from sampling signals is applied in various situations. For reconstruction, a filter based on Shannon's sampling theorem [1] is usually employed. The sampling theorem ensures complete reconstruction of a signal from sampled points whenever the Nyquist condition is satisfied. Violation of the Nyquist condition will result in aliasing phenomena, making it impossible to extract the frequency components that are necessary to reconstruct the signal.

There is a trade-off between the sampling frequency and the high frequency components that can be reconstructed, which influences the quality of the reconstructed signal.

In the case of a musical compact disc(CD), the sampling frequency is set to 44.1 kHz, considering that human audition is not sensitive to frequency components that exceed 22 kHz.

In this paper, we propose a method of signal reconstruction using an adaptive signal reconstruction filter that is implemented with a neural network. The neural network is used to estimate data located at the intermediate points between two existing samples. In general, a trained neural network is required to have a generalization ability, but in our study the neural network is trained specifically to reconstruct a particular signal. therefore, in addition to the sample data, the weights of the neural network have to be provided in the reconstruction process. Consequently, the necessary data size for reconstruction with our method is greater than that for reconstruction with the conventional low-pass filtering method. However, it is expected that the proposed method will have less reconstruction error than the conventional low-pass filtering method, because the conventional method applies minimum assumptions to the signal to be reconstructed, such as the limitation of frequency band width, while the proposed method is specifically trained to reconstruct that signal. Therefore, the proposed method is expected to reduce the total data size in order to reconstruct signal with the same fidelity as the conventional method.

There are some studies that have adopted objectives similar to ours.

Candocia, and Principe[2][3] proposed a method for obtaining a high-resolution image from its low-resolution counterpart, which is called "super-resolution". In this method, the given low-resolution image is clustered into a number of classes, in which subimages with similar local information are grouped as a class. Each class is then linked to a pre-trained neural network called linear associative memory (LAM) for the reconstructon of a high-resolution image based on the particular characteristic of the class. This method uses a group of neural networks that can be used for general images, while our method uses a neural network that is customized to specific data.

Ohira, et al.[4], proposed a reconstruction filter for high-quality image enlargement, establishing a unique sampling function. Although this study and ours have similar objective in generating a reconstruction filter, it differs from ours because it does not deal with missing sample points.

Iwaki, et al. [5], Katagishi, et al. [6], Toraichi, et al. [7], Kamada, et al. [8] proposed a method for a unique sampling function for a reconstruction filter. These studies can be distinguished from ours, because in our research, we propose a method for generating a reconstruction filter that adaptively matches the given signal.

A method using a neural network as the FIR filter was proposed [9] in which the neural network predicts the length of the input points that should be used, which differs from ours in its objective.

In [10], a method of utilizing neural networks for image compression is proposed. The main difference between this research and ours is that, in our proposed research the neural network is used for retrieving omitted sample point data, so that the retrieved data satisfy the Nyquist condition for the original signal.

This paper is organized as follows. In Section 2 the formalization of the problem is explained. Section 3 explains the reconstruction using the neural network. Some typical experimental results are given and discussed in Section 4. The conclusions are given in the final section.

2. Problem Definition

The signal treated in this paper is illustrated in Figure 1. Suppose the original signal $f(t)$ satisfies the following condition:

$$F(\omega)\begin{cases} \neq & 0 \ \ (|\omega| < W) \\ = & 0 \ \ otherwise \end{cases} \tag{1}$$

where $F(\omega)$ is the Fourier transform of the original signal $f(t)$, and W/π is the Nyquist frequency of signal $f(t)$.

$f(t)$ is sampled with the sampling frequency of $1/\tau$ as

$$\frac{1}{\tau} \geq \frac{W}{\pi} \tag{2}$$

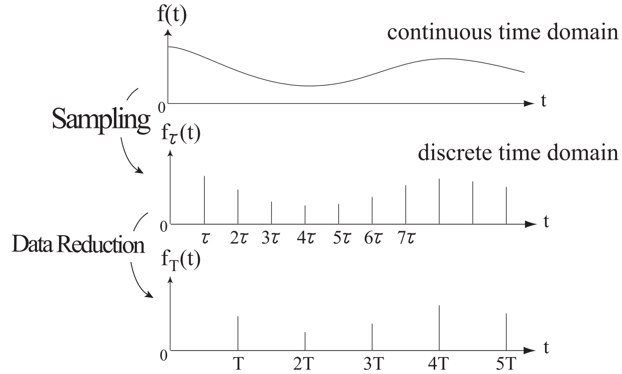


Figure 1: Sampling and data reduction

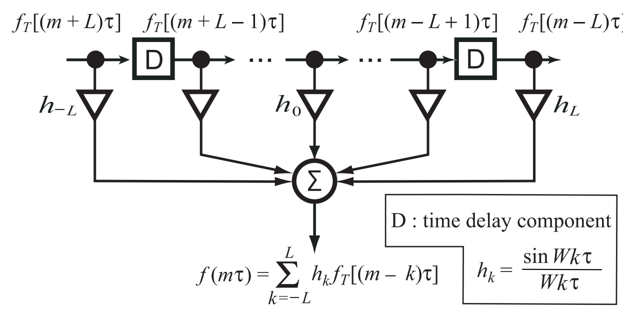


Figure 2: Reconstruction FIR filter based on the sampling theorem

so that we obtain sampled data $f_\tau(t)$ as

$$\begin{aligned} f_\tau(t) &= \sum_{n=0}^{\infty} \delta(t-n\tau)f(t) \\ \delta(t-n\tau) &= \begin{cases} 1 & (t=n\tau) \\ 0 & otherwise \end{cases} \end{aligned} \tag{3}$$

For the purpose of data reduction, we define the sampled data $f_T(t)$ as

$$f_T(t) = \begin{cases} f_\tau(t) & t = mT(m=0,1,2,\cdots) \\ 0 & otherwise \end{cases} \tag{4}$$

where

$$T = N\tau \ \ (N > 1, \ N : integer) \tag{5}$$

f_T will be stored as the sampled data, and used for the reconstruction of the original signal f_τ in the discrete time domain.

Our problem is to obtain a filter implemented with a neural network to reconstruct f_τ from f_T, in the case that T and τ are defined as follows:

$$\frac{1}{\tau} \geq \frac{W}{\pi} \tag{6}$$

$$\frac{1}{T} < \frac{W}{\pi} \tag{7}$$

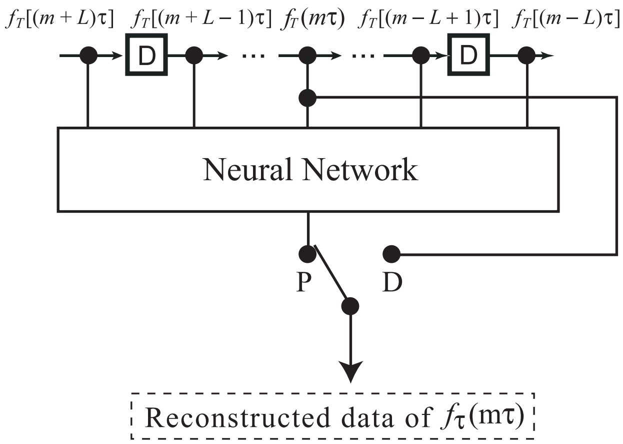


Figure 3: Reconstruction architecture with neural network (When $m\tau \neq NT$, the toggle switch is connected to P, and when $m\tau = NT$, it is connected to D.)

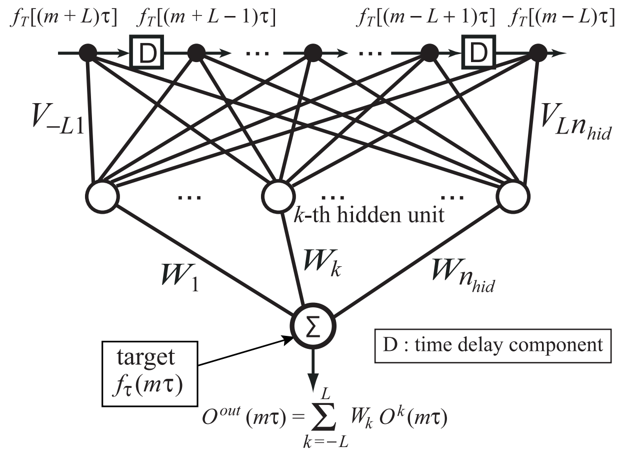


Figure 4: Multilayer perceptron for reconstruction ($O^k(m\tau)$ is the output of k-th hidden unit.)

The neural network is customized for reconstructing a particular signal.

$f(t)$ can always be reconstructed from $f_\tau(t)$ because the condition (2) is satisfied. However, $f_\tau(t)$ can not be reconstructed from $f_T(t)$ because (7) holds. (If (7) does not hold, the FIR filter shown in Figure 2 can reconstruct $f_\tau(t)$ completely.)

3. Neural Network as a Reconstruction Filter

Because $f_T(NT) = f_\tau(NT)$ $(N = 0, 1, 2, \cdots)$ are already available, they do not have to be reconstructed. Hence, we only reconstruct $f_\tau(m\tau)$ $(m = 0, 1, 2, \cdots)$ by the neural network in the case of $m\tau \neq NT$ (Figure 3). It is possible to reconstruct $f_\tau(m\tau)$ by the proposed method when $m\tau = NT$. Then, the performance of the reconstruction is maintained when the linear perceptron is used as a reconstruction filter, but it decreases when a multilayer perceptron (MLP) is used.

When $m\tau \neq NT$, the transversal-type perceptron which has been trained for reconstruction f_τ from f_T is utilized as the filter. We investigated the reconstructing method with a linear perceptron (adaptive linear filter) and MLP with nonlinear hidden units, as shown in Figure 4.

The output of the linear perceptron which has $2L+1$ input units is as follow:

$$O^{out}(m\tau) = \sum_{k=-L}^{L} \omega_k f_T[(m-k)\tau] \quad (8)$$

where ω_k is the connection weight between the $(L-k+1)$-th input unit and the output unit.

The output of MLP with one hidden layer is as follows:

$$\begin{aligned} O^{out}(m\tau) &= \sum_{k=1}^{n_{hid}} W_k O^k(m\tau) \\ O^k(m\tau) &= g(\sum_{j=-L}^{L} V_{jk} f_T[(m-j)\tau]) \\ g(x) &= \frac{1}{1+exp(-\beta x)} \end{aligned} \quad (9)$$

where W_k and V_{jk} are the connection weights between the k-th unit in the hidden layer and the output unit, and the connection weights between the j-th input unit and the k-th unit in the hidden layer, respectively. n_{hid} is the number of hidden units. β is a positive constant. O^k is the output of the k-th hidden unit, while O^{out} is the output of the neural network.

For adapting the neural network to the signal, the weights of the neural network should be corrected so as to minimize the energy function E as follows:

$$E(t_i) = \begin{cases} \frac{1}{2}(O^{out}(t_i\tau) - f_\tau(t_i\tau))^2 & (t_i\tau \neq NT) \\ 0 & (t_i\tau = NT) \end{cases} \quad (10)$$

in which $E(t_i)$ is the error at the t_i-th training iteration. We utilize the steepest descent method. Hence, in the case of the linear perceptron, the connection weights are corrected as follows:

$$\omega_k(t_i+1) = \omega_k(t_i) - \eta \frac{\partial E(t_i)}{\partial \omega_k(t_i)} \quad (11)$$

and in the case of MLP, they are corrected as follows:

$$W_k(t_i+1) = W_k(t_i) - \eta \frac{\partial E(t_i)}{\partial W_k(t_i)} \quad (12)$$

$$V_{jk}(t_i+1) = V_{jk}(t_i) - \eta \frac{\partial E(t_i)}{\partial V_{jk}(t_i)} \quad (13)$$

Table 1: Parameter settings for experiment 1

Parameter	Value
original signal	0.1sec musical signals
T	$n\tau$ $(n = 2, 4, 6, 8)$
τ	1/44.1msec
number of input units	101

Table 2: Parameter settings for experiment 2

Parameter	Value
original signal	0.1sec musical signal
T	$n\tau$ $(n = 4, 6, 8)$
τ	1/44.1msec
number of input units	31-1501

where $\omega_k(t_i)$, $W_k(t_i)$ and $V_{jk}(t_i)$ are the connection weights of the neural network after t_i-th training.

It is expected that $f_\tau(m\tau)$ can be approximated by $O^{out}(m\tau)$.

4. Experiments

In the experiments, two musical signals with a length of 0.1 sec were used. We call them 'Sample 1' and 'Sample 2'. The performance of the proposed method for reconstructing longer signal is explained in Section 4.5. Utilizing the proposed neural network, we attempt to reconstruct the full sample data (the musical data sampled at 44.1 kHz) from the reduced sample points (with the sampling frequency of 44.1/N kHz, $(N > 1)$).

The reconstructions from the reduced sample were carried out by using the linear perceptron and MLP. For comparison, we also reconstructed the data utilizing FIR with the *sinc* response function as follows:

$$\hat{f}(\alpha\tau) = \sum_{j=-1000}^{1000} f_T((\alpha - j)\tau) \frac{\sin \frac{\Omega}{2} j\tau}{\frac{\Omega}{2} j\tau} \qquad (14)$$

where $\hat{f}$ and Ω indicate the reconstructed signal and the cutoff frequency, respectively. For the experiments, the data in the range of $1000 < \alpha < 3000$ are used.

In training the neural network, β in (9) is set to 5 and η in (13) is annealed from 0.1 to 0.00001.

The error R_E between the original and reconstructed signals is calculated as follows:

$$R_E = \frac{1}{N_{data}} \sum_{\alpha} (\hat{f}(\alpha\tau) - f_\tau(\alpha\tau))^2 \qquad (15)$$

where N_{data} is the number of reconstructed data that is the actual sample number of the reconstructed signal.

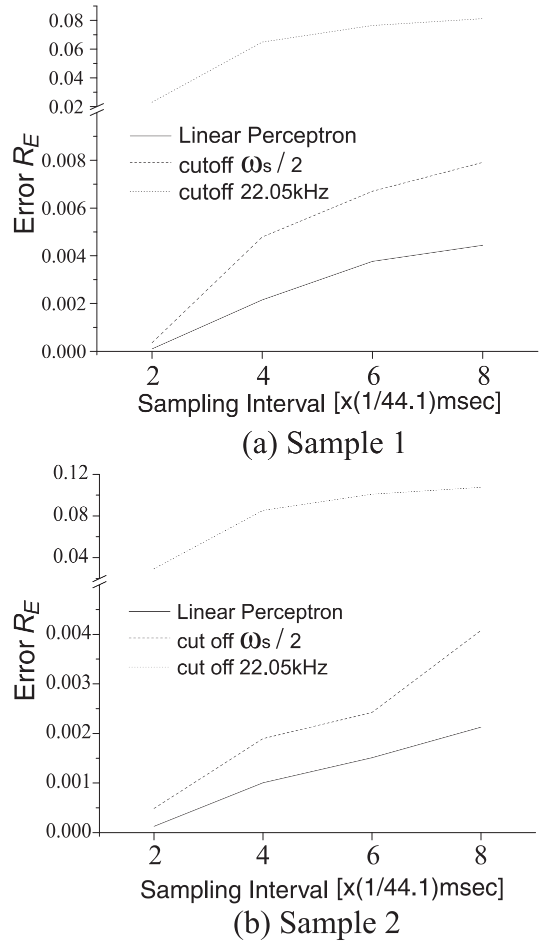


Figure 5: Reconstruction error of the linear perceptron vs sampling interval

For example, if the length and the sampling frequency of a reconstructed signal are just 1 sec and 44.1 kHz, respectively, N_{data} is 44100. Values of f_τ and f_T are normalized into the range of [-1,1].

4.1 Performance of Linear Perceptron

We compared the performance of the proposed linear perceptron with that of the conventional FIR filter (14).

Figure 5 shows the performance of the respective reconstruction methods. The parameter settings are shown in Table 1. 'cutoff $\omega_s/2$' and 'cutoff 22.05kHz' indicate the reconstruction error of $\hat{f}$ when $\Omega = \omega_s$ and $\Omega/2\pi = 44.1kHz$, respectively, where $\omega_s = 2\pi/T$. The result in Figure 5 shows that the linear perceptron performed better.

In experiment 2, we reconstructed the signal with the linear perceptron with regard to various numbers of input units when the sampling intervals were 4/44.1 msec, 6/44.1 msec and 8/44.1 msec.

Figure 6 shows the result of the experiment. The

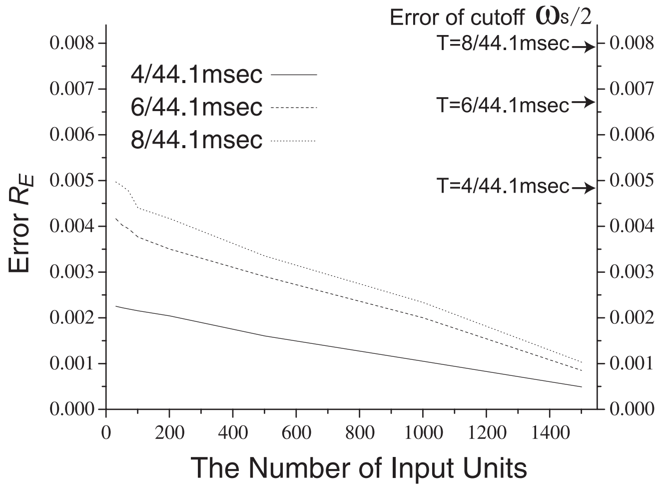


Figure 6: Reconstruction error vs the number of input units of the linear perceptron

arrows in this figure show the error of the conventional FIR filter ($\Omega = \omega_s$) with the same input data. It is shown that even if the linear perceptron has only 31 input units, it performs better than the conventional FIR filter, and the error decreases as the number of input units increases.

4.2 Performance of MLP

We compared the performances of MLP and the linear perceptron. In the experiment below, MLPs with fixed input units of 101 and 10, 20 and 40 hidden units were used, and the other parameter settings are the same as those of experiment 1. In Figure 7, the numbers inside the brackets show the respective unit numbers in the MLP's input, hidden and output layers.

Figure 7 shows that the reconstruction error of MLP is less than that of the linear perceptron, and the error decreases as the number of hidden units of MLP increases.

4.3 Estimation of Data Compression Rate

In the previous subsection, we did not consider the amount of data necessary for reconstruction. Because in the proposed method the weights of the neural network are required in addition to the sampled data f_T for reconstruction, we have to analyze the relation between the amount of data and the reconstruction fidelity in order to compare the performances of the proposed method and the conventional low-pass filtering method.

Because $f_T\left([m + N(\frac{T}{\tau})]\tau\right) = 0$ $(m\tau \neq NT)$, in the case of the linear perceptron, the values of $\omega_{\pm n\frac{T}{\tau}}$ $(n = 0, 1, 2, \cdots)$ are not required, and in the case of MLP,

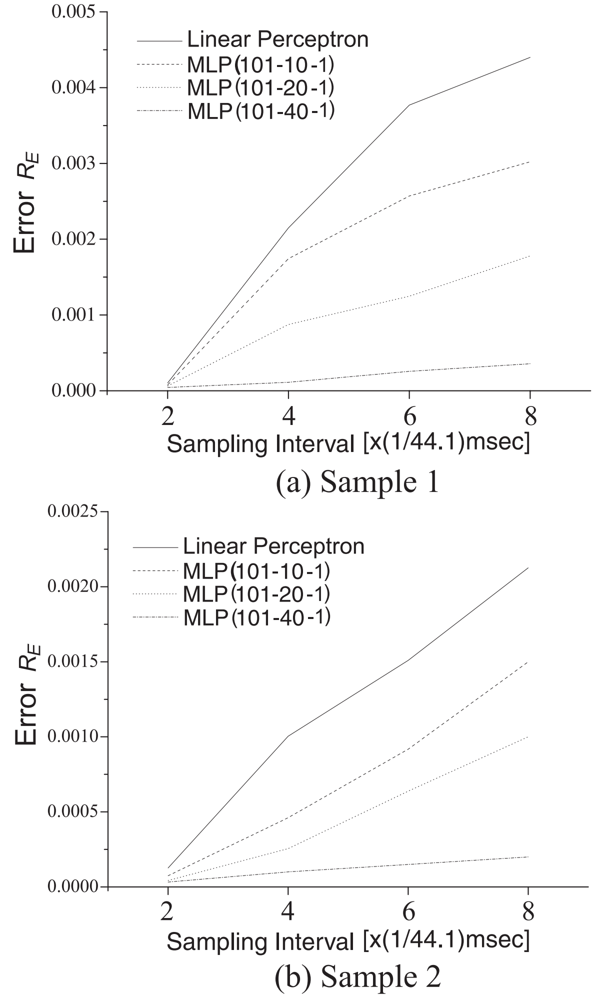


Figure 7: Reconstruction error of MLP vs sampling interval.

the values of $V_{\pm n\frac{T}{\tau}\ i}$ $(n = 0, 1, 2, \cdots; i = 0, 1, \cdots, n_{hid})$ are not required. Therefore, in the case of the linear perceptron, the number of weight data required for reconstruction is as follow:

$$2L - \left[\frac{L-r}{(T/\tau)} \times 2\right] \quad (16)$$

and in the case of MLP, it is as follow:

$$\left(2L - \left[\frac{L-r}{(T/\tau)} \times 2\right] + 1\right) \times n_{hid} \quad (17)$$

where $2L+1$ is the number of input units of the neural network, r is the remainder of the division of L with T/τ, and n_{hid} is the number of hidden units of MLP. The sum of the number of sample points of f_T and (16) or (17) is called 'data size' in the following discussions.

Figure 8 shows the relation between the data size and the reconstruction error.

Figure 8 shows the superiority of the linear perceptron to the conventional FIR filter with regard to

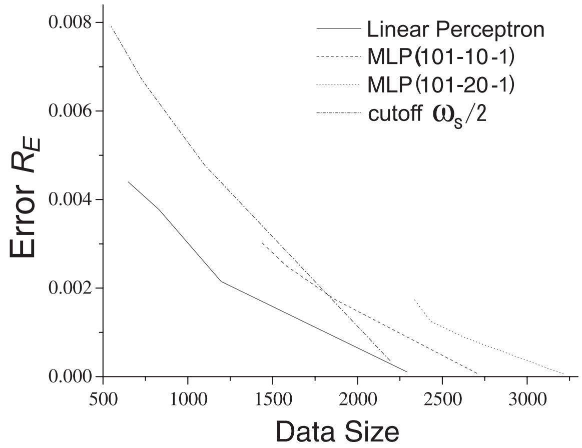


Figure 8: The reconstruction error vs data size

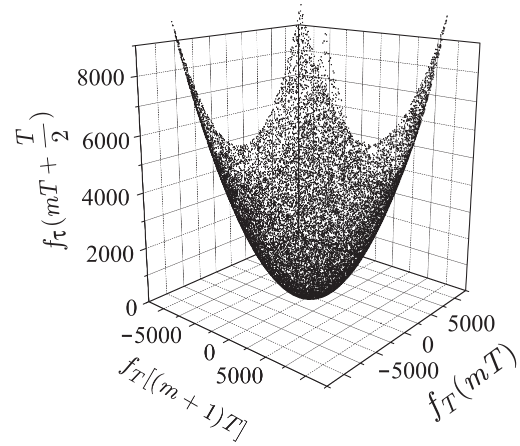


Figure 9: Distribution of nonlinear data ($T = 2\tau$)

the data size (compression rate) and the reconstruction quality. The linear perceptron requires a smaller number of data than the conventional FIR filter to reconstruct a signal with similar fidelity. For example, the linear perceptron which has only 31 input units can reconstruct the 44.1kHz-sampled data from the 5.52kHz-sampled data with approximately the same reconstruction error as the FIR filter has for reconstructing the signal from the 11.025kHz-sampled data, as shown in Figure 6. In this case, the data size for the linear perceptron is $552 + 24$, while the FIR filter requires 1102 data units.

In the experiment, MLP may seem to be inferior to the linear perceptron as shown in Figure 8, but the error of the linear perceptron cannot be less than that of MLP even if the linear perceptron has a large number of input units as shown in Figures 6 and 7.

4.4 Data Structure

When the reconstruction with the perceptron which has $2L + 1$ input units is carried out, the input signal and the target signal can be written as follows:

$$\begin{cases} \text{input} & \big(f_T[(m+L)\tau], f_T[(m+L-1)\tau], \cdots, \\ & \quad f_T[(m-L+1)\tau], f_T[(m-L)\tau]\big) \\ \text{target} & f_\tau(m\tau) \end{cases} \tag{18}$$

$$(m = L, L+1, L+2, \cdots)$$

Therefore, a training sample can be expressed as a point in $(2L + 2)$-dimensional space as

$$\big(f_T[(m+L)\tau], f_T[(m+L-1)\tau], \cdots, \\ f_T[(m-L+1)\tau], f_T[(m-L)\tau)], f_\tau(m\tau)\big) \tag{19}$$

Consider a case of reconstructing the data at $mT + \frac{T}{2}$ from the sampled data at mT and $(m + 1)T$. Then the number of inputs is $\frac{T}{\tau} + 1$, assuming that $\frac{T}{\tau}$ is an even number, and (19) can be rewritten as follows,

$$\Big(f_T(mT), 0, 0, \cdots, 0, 0, f_T[(m+1)T], f_\tau(mT + \tfrac{T}{2})\Big) \\ (m = L, L+1, L+2, \cdots) \tag{20}$$

which defines essentially three-dimensional space as

$$\left(f_T(mT), f_T[(m+1)T], f_\tau(mT + \frac{T}{2})\right) \tag{21}$$

Thus, for the above training point, the neural network has to provide a mapping from $(f_T(mT), f_T[(m+1)T])$ to $f_\tau(mT + \frac{T}{2})$.

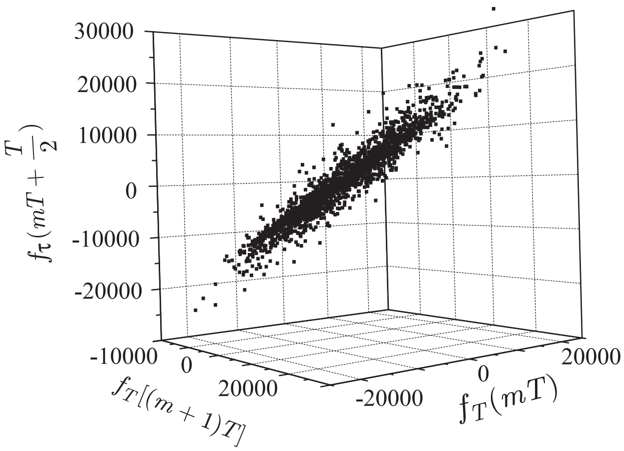
We have confirmed that for some cases the MLP can achieve significantly better compression rates than the linear perceptron.

Table 3: Reconstruction error from Fig. 9 data

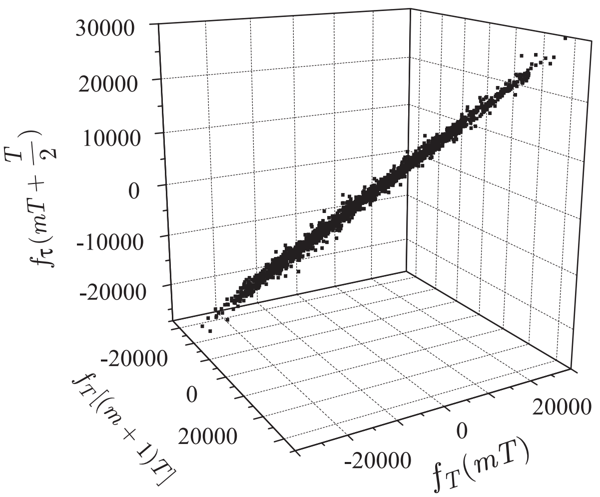
	L.P.	MLP	
hidden units	—	5	10
error	3.36×10^{-3}	1.1×10^{-5}	2.0×10^{-6}

Table 3 shows that when the signal is plotted according to (21) as shown in Figure 9, the reconstruction error of MLP is significantly less than that of the linear perceptron. Then, the data compression rate of MLP is clearly superior to the linear perceptron.

The plots of the musical signal discribed in Section 4 according to (21) are shown in Figures 10 and 11. In these cases, sample points are distributed around a linear plane perturbed by random noises. Because to some extent the distribution's linearity can be assumed in this example, the linear perceptron works well.

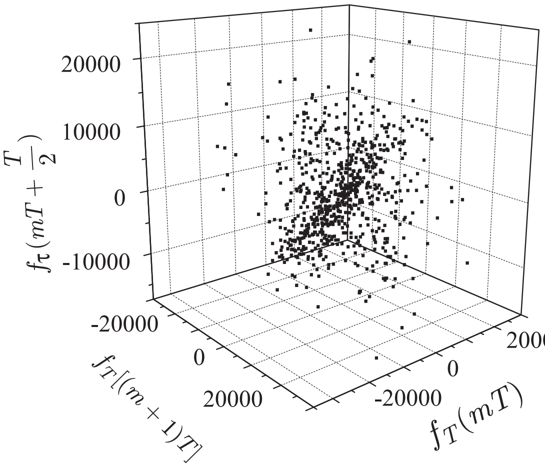


(a) Sample 1

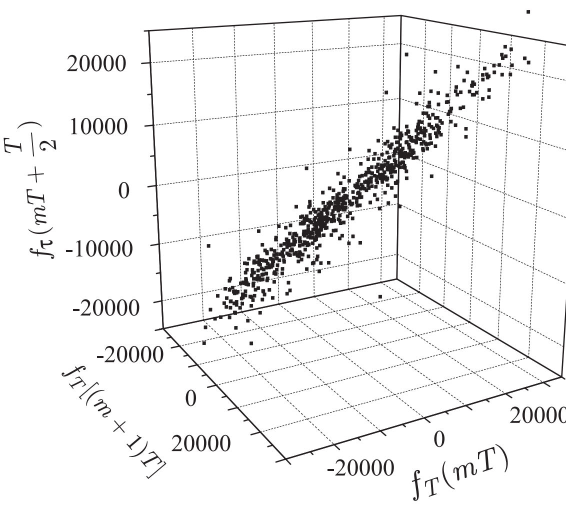


(b) Sample 2

Figure 10: Distributions of samples which are used for experiments ($T = 2\tau$)



(a) Sample 1



(b) Sample 2

Figure 11: Distributions of samples which are used for experiments ($T = 6\tau$)

4.5 Performance on Longer Data

In the previous subsections, we have investigated the characteristic of the musical signals sampled with various intervals and the performance of the proposed model. With consideration of putting the proposed model into practical usage, we also tested the proposed model with longer musical signals. We used twenty-one 0.1 sec signals, three 1 min signals, and a 3 min signal. The 1min signals were obtained by dividing the 3 min signal into three, and from each of these, seven 0.1 sec signals are taken randomly. The parameter settings for the experiment are given in Table 4. The weights of the neural network are fixed as trained ones during the signal reconstruction process.

The average performances of the proposed model regarding the musical signals with three different lengths are shown in Figure 12. It is obvious that the proposed model performs significantly better than the conventional FIR filter. The reconstruction error of the average of the 0.1 sec signals and that of the 3 min signal are almost the same. It is expected that as the signal length becomes longer, the reconstruction error increases because the number of data sets increases.

5. Conclusion

We have proposed an adaptive signal reconstruction filter which uses a neural network to deal with sparse data sampling that does not satisfy the Nyquist condition. The qualitative and quantitative performance evaluations show that the proposed method

Table 4: Parameter settings for the experiment in Section 4.5

Parameter	Value
original signal	0.1 sec, 1 min, and 3 min musical signal
T	4τ
τ	1/44.1 msec
number of input units	101

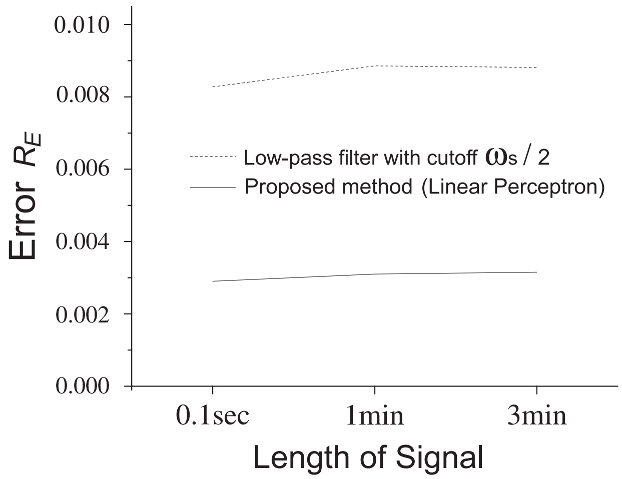


Figure 12: Reconstruction error vs length of signal

is better than the conventional low-pass filtering method.

Our future work is to examine the performance of the proposed method with respect to the spectral structure, dynamic range and stationarity of the target signal. Although we used a simple linear perceptron and a three-layer perceptron in the experiments, the required complexity of the neural network is another future work. We expect that the proposed reconstruction method will be introduced to various applications, such as super-resolution imaging and super audio. Another possibility is to combine the proposed method with existing data compression methods such as MP3 and ATRAC.

References

[1] C. E. Shannon: Communication in the presence of noise, Proc. IRE, Vol. 137, No. 1, pp.10-21, 1949.

[2] F. M. Candocia and J. C. Principe: Super-resolution of images based on local correlations, IEEE Trans. Neural Networks, Vol. 10, No. 2, pp.372-380, 1999.

[3] F. M. Candocia and J. C. Principe: A neural implementation of interpolation with a family of Kernels, Proc. Int. Conf. Neural Network, Vol.3, pp.372-380, 1997.

[4] M. Ohira, K. Mori, M. Obata, K. Wada and K. Toraichi: High quality image enlargement and reconstruction by adaptive two-dimensional sampling theorem, The Journal of IIEE of Japan, Vol. 28, No. 5, pp.620-627, 1999.

[5] M. Iwaki and K. Toraichi: Sampling theorem for spline signal space of arbitrary degree, IEICE Trans. Fundamentals, Vol. E77-A, No. 5, pp.810-817, 1994.

[6] K. Katagishi, K. Toraichi, M. Obata and K. Wada: A practical least squares approximation based on biorthonormal expansions in the signal space of piecewise polynomials, Trans. of IEE Japan, Vol. 118, No. 3, pp.353-365, 1998.

[7] K. Toraichi, M. Kamada and R. Mori: A quadratic spline function generator, IEEE Trans. Acoust., Speech & Signal Process., Vol. ASSP-37, No.4, pp.534-544, 1989.

[8] M. Kamada, K. Toraichi and R. Mori: Sampling bases for the signal spaces composed of spline functions, Trans. IEICE, Vol. J71-A, No.3, pp.875-881, 1988.

[9] S. Kaleta, D. Novotny and P. Sincak: Prediction systems based on FIR BP neural networks, Artificial Neural Networks - ICANN 2001, Lecture Notes in Computer Science, Springer, Vol. 2130, pp.725-730, 2001.

[10] M. I. H. Bhuiyan, M. K. Hasan, N. C. Hammadi and T. Yahagi: Image compression with neural network using dynamical construction algorithm, Journal of Signal Processing, Vol. 5, No. 6, pp.445-454, 2001.